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UNBIASEDNESS AND LEAST ABSOLUTE ERROR ESTIMATION

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SUMMARY

Taylor [4] proposed a linear programming formulation of the least absolute error estimation problem and claimed unbiasedness of the estimate in case of errors distributed symmetrically around zero. Here some drawback in Taylor's approach has been pointed out and a procedure suggested to incorporate the condition of unbiasedness in the LP formulation to yield an estimate that minimises the sum of absolute errors in the class of unbiased estimates. An example has been given to illustrate the procedure.

Keywords : Least absolute error estimation; linear programming, generalised inverse.

Introduction

Recently, least absolute error (LAE) estimation has received much attention as an alternative to least squares (LS) estimation, primarily owing to its insensitivity to outliers. The least squares estimate may be spoilt by a single grossly outlying observation. In fact, LS estimators are far from optimal in many non-Gaussian situations. The LAE estimators are maximum likelihood and hence asymptotically unbiased and efficient when the errors follow a Laplace distribution. A state-of-the-art survey of LAE estimation appears in Narula and Wellington [1].

LAE estimators in general need not be unbiased. Sielken and Hartley [3] as well as Taylor [4] have proposed linear programming (LP) solutions to the LAE estimation problem and have proved the unbiasedness of the

LAE estimators. The algorithm given by the former leads to an unbiased estimate only on the tacit assumption of a unique solution to the LP problem.

The present work points out some drawback in Taylor's approach (section 2) and suggests a procedure to incorporate the condition of unbiasedness in the LP formulation to yield an estimator that minimises the sum of absolute errors among unbiased estimators. An example has been given in Section 4 to illustrate the procedure suggested here.

2. Taylor's Method Reviewed

Consider the linear model $y = X\beta + u$, where y is in $n \times 1$ vector of response variables, X is the given $n \times p$ incidence matrix (of regressor or predictor variables), β is the $p \times 1$ vector of unknown parameters and u is the $n \times 1$ vector of random errors.

Taylor claims that the optimal solution $(\hat{\beta}, \hat{u})$ to the LAE estimation problem formulated as an LP problem is given by

$$\begin{matrix} y_{(1)} \\ y_{(2)} \end{matrix} = \begin{pmatrix} X_{(1)} & 0 \\ X_{(2)} & I \end{pmatrix} \begin{pmatrix} \hat{\beta} \\ \hat{u} \end{pmatrix} \tag{2.1}$$

where $y_{(1)}$ and $y_{(2)}$ are sub-vectors of orders $p \times 1$ and $\overline{n-p} \times 1$ respectively, $X_{(1)}$ and $X_{(2)}$ being sub-matrices of orderss $p \times p$ and $\overline{n-p} \times p \cdot I$ is the $\overline{n-p} \times n$ unit matrix and 0 is $p \times n$ null matrix.

Let H be the total sample space of u. It was proved that if u is symmetrical about origin, then unbiasedness of LAE estimates follows for a given subset H_i satisfying the relation

$$-\bar{I} X_{(2)} X_{(1)}^{-1} u_{(1)} + \bar{I} u_{(2)} \geqslant 0$$

where $\bar{I}$ is a diagonal matrix with elements $+1$ or -1. Taylor then took symmetry of u as a sufficient condition for the LAE method to provide an unbiased estimate. According to Taylor, the LP problem necessarily has a feasible solution and being bounded below, has an optimum basic feasible solution. But an optimum (basic feasible) solution need not be of the form (2.1). In fact, there may not be any basic feasible solution of the form (2.1) when X does not have full rank. It seems that even in the full rank case there may not be an optimal solution of the form (2.1).

Finally, unbiasedness of LAE estimates, when we confine ourselves to H_i (assuming it not to be void), can not lead to unbiasedness of the estimators in H. Firstly the H_i's defined by Taylor are not mutually exclusive and numerous examples can be constructed where H_i's do not exhaust the whole sample space H. The first objection is not so serious, because the set of H_i's can be divided into a set of mutually exclusive sets and it can be shown, following the same line of proof as that of Taylor, that confined to each of these mutually exclusive subspaces of H, the estimators are conditionally unbiased. What is more serious is that UH_i may not be H. Then the estimator is not defined at all in $H - (UH_i)$ and as such unbiasedness does not follow.

3. Unbiased LAE Estimation

Our problem is to minimise $e'u^+$

$$\text{subject to } X\beta + u = y \tag{3.1}$$

where β and u are unrestricted in sign and u^+ denotes the column vector of absolute u-values. e is a vector of unities

Let $\hat{\beta} = By$ and u' denote some

solution of (3.1). We know that any linear parametric function of the form $C'X\beta$ is estimable. As an estimate of $C'X\beta$ we propose $C'X\hat{\beta}$. Now in order that $E(C'X\beta) = C'X\beta$ it is necessary and sufficient that $E(C'\hat{u}) = 0$ for all C and all β, assuming $E(u) = 0$. This implies $E(\hat{u}) = 0$ for all β. Thus u should belong to the error space, a necessary and sufficient condition for which is $u = Dy$ where $DX = 0$. From this we can write $D = Z [I - X\bar{X}]$ where Z is arbitrary.

Since $X\hat{\beta} + \hat{u} = y$, we have

$$XBy + Z(I - X\bar{X})\, y = Iy$$

or $$Z(I - X\bar{X})\, y = (I - XB)\, y \quad \text{for all } y \text{ and } Z.$$

Thus B is a g-inverse of X. Hence the problem of LAE estimation incorporating unbiasedness be reformulated as the minimisation of $Z = e'\{(I - XB)\, y\}^+$ where B is a g-inverse of X, a general expression of which is $B = \bar{X} + U - \bar{X}XUX\bar{X}$ where U is arbitrary. So the problem reduces to the minimisation of $Z = e'\{(I - XU)(I - X\bar{X})\, y\}^+$. To solve this LP problem for U, we shall first find out a particular $\bar{X}$ for a given X, then define $v \geqslant 0$; $w \geqslant 0$ such that,

$$(I - X'U)(I - X\bar{X})\, y = v - w \tag{3.2}$$

and minimise $e'(v + w)$ subject to (3.2). The unbiased LAE estimator will simply be

$$\hat{\beta} = B \cdot y = \{\bar{X} + U - \bar{X}XUX\bar{X}\}\, y. \tag{3.3}$$

4. A Simulation Study

To study the behaviour of LAE estimates obtained by the method suggested here, a simulation exercise was carried out. For simplicity, the following two-input model was considered

$$y = \beta_1 x_1 + \beta_2 x_2 + u$$

and simulation started with a fixed 10×2 input matrix X given by

$$X' = \begin{pmatrix} 12 & 16 & 20 & 18 & 23 & 25 & 19 & 18 & 15 & 22 \\ 6 & 10 & 12 & 9 & 7 & 10 & 8 & 9 & 15 & 13 \end{pmatrix}$$

Three models were considered, specifying the distribution of u as

(i) normal with mean 0 and Sd. 0.2

(ii) rectangular with p.d.f. $f(u) = \dfrac{1}{2\sqrt{3.33}}$, $-3.33 \leqslant u \leqslant 3.33$.

(iii) type II (extreme valued) with p.d.f.

$$f(u) = y_0 \left(1 - \frac{u^2}{a^2}\right)^m, \ -\infty < u < \infty, \ m = -0.5, \ a^2 = 0.08.$$

Parameters of each distribution were so specified as to yield a variance of 0.04 for each distribution in order that results become comparable.

For a single solution in a specific model, random values of u were selected from the corresponding distributions, then Y values were generated by taking, $\beta_1 = 0.6$ and $\beta_2 = 0.1$. For each model 30 such solutions from different sets of u values were obtained.

The algorithm for the LP solution in each model is as follows :

For an input matrix X we find a g-inverse $\bar{X}$
Then the problem is to minimise $Z = \Sigma v_i + \Sigma w_i$
given $(I - X\bar{X})\, y = X'U\,(I - X\bar{X})\, y + v - w$ (4.1)
where $u = (u_{ij})$ is unrestricted in sign, $v \geqslant 0$, $w > 0$.

Defining $u_{ij} = t_{ij} - s_{ij}$, $t_{ij} \geqslant 0$ $s_{ij} \geqslant 0$, $i = 1, 2$, $j = 1, 2 \ldots 10$ (4.1) can be written as

$$(I - X\bar{X})\, y = X\,(T - S)\,(I - X\bar{X})\, y + v - w$$

where $T \geqslant 0$, $S \geqslant 0$, $v \geqslant 0$, $w \geqslant 0$.

The total number of variables in the problem is 60.

As initial basic feasible solution, we start with

$$t_{ij} = 0, \quad s_{ij} = 0 \quad \text{for all } i \text{ and } j$$

and if $(I - X\bar{X})\, y \geqslant 0$, $v_i = (I - X\bar{X})\, y$ and $w_i = 0$

while if $(I - X\bar{X})\, y < 0$, $v_i = 0$, $w = -(I - X\bar{X})\, y$.

From solutions of U, the β values are obtained from (3.3).

The average values of 30 solutions for each of β_1 and β_2 in 3 models, together with respective mean squares are presented below :

ESTIMATED VALUES OF PARAMETERS WITH MSE IN 3 MODELS

Model	$\hat{\beta}_1$		$\hat{\beta}_2$	
	Mean	*MSE*	*Mean*	*MSE*
Normal	0.6002	0.0153	0.0979	0.0283
Rectangular	0.5952	0.0141	0.1035	0.0282
Type II	0.5954	0.0132	0.1052	0.0235

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